MTH849 QUALIFYING EXAM - AUGUST 2021

1. Instructions

There are five problems on this exam. Complete as many problems as possible. Your four highest scoring answers will be used to determine your grade on the exam. A preference in scoring will be given to complete answers to entire problems, in contrast to partial answers to possibly more problems. You have also been given a pdf document containing various theorems and definitions for your use on this exam.

2. The exam questions

Problem 1. Assume that $\Omega \subset \mathbb{R}^n$ is open, connected, and has a C^1 boundary. Further assume that $\Gamma \subset \partial\Omega$, and $|\Gamma| > 0$. Define the subspace, V, as

$$V = \{v \in H^1(\Omega) \ : \ Tv|_\Gamma = 0\},$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

(i) Prove that there exists and constant, C, depending on Ω and n, so that

$$\forall \ v \in V, \quad \int_\Omega v^2 dx \le C \int_\Omega |\nabla v|^2 \, dx.$$

(ii) Identify in which part of your proof you have used the assumption that Ω is connected.
(iii) Identify in which part of your proof you have used the assumption that Ω has a C^1 boundary.

Problem 2. The following is a mixed Nuemann-Dirichlet boundary value problem. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, and whose boundary is C^1. Assume that there are two subsets, $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \subset \partial\Omega$ so that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, and that $|\Gamma_1| > 0$, $|\Gamma_2| > 0$. The equation is

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \subset \partial\Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \Gamma_2 \subset \partial\Omega. \end{cases} \tag{1}$$

Define the subspace of H^1, V, as

$$V = \{v \in H^1(\Omega) \ : \ Tv|_{\Gamma_1} = 0\}.$$

(i) Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, that u is a classical solution of (1), and that $v \in C^1(\overline{\Omega}) \cap V$, prove that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\Gamma_2} g v dS.$$

(ii) Prove that there exists a unique function, $u \in V$, that satisfies as a weak definition of solution to (1), the following requirement

$$\forall\ v \in V, \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\Gamma_2} gTv dS. \tag{2}$$

You may use Problem 1, if you would like.

Problem 3. Prove that if $\Omega \subset \mathbb{R}^n$ is open and bounded, $b \in C^0(\Omega; \mathbb{R}^n)$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of

$$-\Delta u + b(x) \cdot \nabla u \leq 0 \quad \text{in} \quad \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u.$$

Problem 4. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, and has a C^1 boundary. The following is the Neumann problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

(i) Define V to be the subspace of H^1 as

$$V = \{v \in H^1(\Omega) \ : \ \int_\Omega v dx = 0\}. \tag{4}$$

Prove that for all $f \in L^2(\Omega)$ and all $g \in L^2(\partial\Omega)$, there exists a unique $v \in V$ that satisfies the following equation:

$$\forall\ w \in V, \quad \int_\Omega \nabla v \cdot \nabla w dx = \int_\Omega f w dx + \int_{\partial\Omega} gTw dS, \tag{5}$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

(ii) The actual definition of weak solution that is compatible with classical solutions of (3) is that $u \in V$, and

$$\forall\ w \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla w dx = \int_\Omega f w dx + \int_{\partial\Omega} gTw dS, \tag{6}$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

(a) Identify a restriction on $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ so that given any f and g that satisfy your restriction, there exists a unique $u \in V$ that solves (6).

(b) Furthermore, prove that given such an f and g, there does indeed exists a unique $u \in V$ that solves (6).

If you do not complete part (ii), you will not be given credit for this problem.

Problem 5. Assume each of the following equations admits at least one weak solution– defined in the appropriate sense given below– and call it u_0. For each equation, say whether or not the solution is unique. If the solution is unique, then prove why. If the solution is not unique, then give an example.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected set, with a C^1 boundary, and that $\Gamma \subset \partial\Omega$ and $|\Gamma|_{\partial\Omega} > 0$.

(i)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H_0^1(\Omega)$ satisfies

$$\forall\ v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

(ii)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \Gamma \subset \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_\Gamma T v dS.$$

(iii) for $c_0 > 0$,

$$\begin{cases} -\Delta u + c_0 u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \Gamma \subset \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c_0 u v dx = \int_\Omega f v dx + \int_\Gamma T v dS.$$

(iv)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \text{no boundary condition given} & \end{cases}$$

Here, we say a weak $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

(The boundary conditions can be ignored because for the test functions, $Tv = 0$.)

(v) for $c_0 > 0$,

$$\begin{cases} -\Delta u + c_0 u = f & \text{in } \Omega \\ \text{no boundary condition given} & \end{cases}$$

Here, we say a weak $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c_0 u v dx = \int_\Omega f v dx.$$

(The boundary conditions can be ignored because for the test functions, $Tv = 0$.)

As a hint to all parts, one may consider the equation solved in the weak sense by the difference of any two possible solutions.